Dynamics of general relativistic spherically symmetric dust thick shells

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Abstract

We consider a spherical thick shell immersed in two different spherically symmetric space-times. Using the fact that the boundaries of the thick shell with two embedding space-times must be nonsingular hypersurfaces, we develop a scheme to obtain the underlying equation of motion for the thick shell in general. As a simple example, the equation of motion of a spherical dustlike shell in vacuum is obtained. To compare our formalism with the thin shell one, the dynamical equation of motion of the thick shell is then expanded to the first order of its thickness. It is easily seen that the thin shell limit of our dynamical equation is exactly that given in the literature for the dynamics of a thin shell. It turns out that the effect of thickness is to speed up the collapse of the shell.

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I. Introduction

The thin shell formalism of general relativity has found wide applications in general relativity and cosmology[1-3]. Studies on gravitational collapse, dynamics of bubbles and domain walls in inflationary models, wormholes, signature changes, structure and dynamics of voids in the large scale structure of the universe are some of the applications. Thin shells are considered as idealized zero thickness objects, with a δ-function singularity in their energy-momentum and Einstein tensors. This is regarded to be an idealization of a real shell with a finite thickness. However, the dynamics of a real thick shell has been rarely discussed in the literature because of the complexity one is faced with when trying to define it within general relativity and to find its exact underlying dynamical equations. The outstanding paper that modifies the Israel thin shell equations to treat the motion of spherical and planar thick domain walls is that of Garfinkle and Gregory[4]. Their work deals with an expansion of the coupled Einstein-scalar equations in powers of the thickness of the domain wall (see also [5]). According to the results of that paper, the effect of thickness in the first approximation is to reduce effectively the energy density of the wall compared to the corresponding thin domain wall, and therefore to increase the collapse velocity of the wall in vacuum.
In this paper we first generally suggest the proper matching conditions on the boundaries of the spherical thick shell embedded in an inner and an outer spherically symmetric space-time. As a simple example, these matching conditions are then used to investigate the motion of spherical dustlike thick shell in vacuum.

II. The junction conditions

Consider a spherically symmetric thick shell with two boundaries Σ_1 and Σ_2 dividing the space-time into three-regions: $\mathcal{M}_{in}$ for inside the inner boundary $\Sigma_1, \mathcal{M}_{out}$ for outside the outer boundary Σ_2, and $\mathcal{M}$ for the thick shell having two boundaries Σ_1 and Σ_2. First of all, let us write down the appropriate junction condition on each boundary Σ_j (j=1,2) treated as a (2+1)-dimensional timelike hypersurface. We expect the continuity of the second fundamental form of Σ_j, or the extrinsic curvature tensor K_{ab} of Σ_j, so that we can consider $\Sigma_1(\Sigma_2)$ as a boundary surface separating $\mathcal{M}$ region from $\mathcal{M}_{in}$ ($\mathcal{M}_{out}$). This crucial requirement is formulated as

$$[K_{ab}] \stackrel{\Sigma_j}{=} 0 \qquad\qquad (j = 1, 2), \tag{1}$$

where the square bracket indicates the jump of K_{ab} across Σ_j, Latin indices range over the intrinsic coordinates of Σ_j denoted by $(\tau_j, \theta, \varphi)$, where τ_j is the proper time of Σ_j. In particular, the angular component of Eq. (1) on each boundary is written as

$$K_\theta^{\theta +}\Big|_{\Sigma_1} - K_\theta^{\theta -}\Big|_{\Sigma_1} = 0, \tag{2}$$

$$K_\theta^{\theta +}\Big|_{\Sigma_2} - K_\theta^{\theta -}\Big|_{\Sigma_2} = 0, \tag{3}$$

where the superscript $+(-)$ refers to the side of Σ_j towards which the corresponding unit spacelike normal vector $n^\alpha(-n^\alpha)$ points. This means that on $\Sigma_1(\Sigma_2)$, the superscript $+$ refers to the region $\mathcal{M}(\mathcal{M}_{out})$ and the superscript $-$ refers to the region $\mathcal{M}_{in}(\mathcal{M})$. Adding Eqs. (2) and (3), we obtain equation

$$K_\theta^{\theta+}\Big|_{\Sigma_2} - K_\theta^{\theta-}\Big|_{\Sigma_1} + K_\theta^{\theta+}\Big|_{\Sigma_1} - K_\theta^{\theta-}\Big|_{\Sigma_2} = 0. \tag{4}$$

In the next section we will apply the general equation (4) to the special case of a collapsing spherical dust shell in vacuum.

III. Collapse of a spherical dust thick shell in vacuum

Consider the Lemaitre-Tolman-Bondi (LTB) metric to describe the dust thick shell. In the synchronized comoving coordinates $(\tau, r, \theta, \varphi)$ the metric is written in the form [6]

$$ds^2 = -d\tau^2 + \frac{R'^2}{1+E(r)}dr^2 + R^2(r,\tau)(d\theta^2 + \sin^2\theta d\varphi^2), \tag{5}$$

where overdot and prime denote partial differentiation with respect to τ and r, respectively, and $E(r)$ is an arbitrary real function such that $E(r) > -1$. Then the corresponding Einstein field equations turn out to be

$$\dot{R}^2(r,\tau) = E(r) + \frac{F(r)}{R}, \tag{6}$$

$$8\pi G\rho(r,\tau) = \frac{F'(r)}{R^2 R'}, \tag{7}$$

where $\rho(r,\tau)$ is the energy density of the matter fluid in $\mathcal{M}$, and $F(r)$ is another arbitrary real smooth function such that $F(r) > 0$. Furthermore, in order to avoid shell crossing of dust matter during their radial motion, we require $R'(r,\tau) > 0$. This together with the assumption of positive mass density $\rho(r,\tau) > 0$, implies that $F'(r) \geq 0$. The induced intrinsic metric on Σ_j may be represented as

$$ds^2\Big|_{\Sigma_j} = -d\tau_j^2 + R_j^2(\tau_j)(d\theta^2 + \sin^2\theta d\varphi^2) \qquad\qquad (j = 1, 2), \tag{8}$$

where $R_j(\tau_j)$ being the proper radius of Σ_j. For simplicity, we may assume that the boundaries Σ_1 and Σ_2 are comoving with respect to the LTB geometry. This requires that the peculiar velocity of Σ_j measured by the comoving observers of $\mathcal{M}$ to be zero. Then the matching relations yield

$$\tau = \tau_j + \text{constant} \qquad\qquad (i = 1, 2). \tag{9}$$

Now, we may define the constant comoving thickness of the shell as follows

$$2\delta = r_2 - r_1, \tag{10}$$

where r_1 and r_2 are comoving radii of the boundaries Σ_1 and Σ_2, respectively. We assume now the spherical dust thick shell to be immersed in vacuum. In this case, the space-time exterior to the shell is Schwarzschild, and the interior is taken to be Minkowski flat space-time. Different terms appearing in the equation (4) may now be explicitly derived. Using the metric (5) together with Eqs. (6), (7) and (9), we compute the relevant extrinsic curvature tensors in the region $\mathcal{M}$ as

$$K_\theta^{\theta+}\Big|_{\Sigma_1} = \frac{1}{R_1}\sqrt{1+\dot{R}_1^2 - \frac{F(r_1)}{R_1}} \quad , \quad K_\theta^{\theta-}\Big|_{\Sigma_2} = \frac{1}{R_2}\sqrt{1+\dot{R}_2^2 - \frac{F(r_2)}{R_2}} \quad , \tag{11}$$

where $R_j \equiv R(r_j, \tau)$. Furthermore, the following expressions for the relevant extrinsic curvature tensors in $\mathcal{M}_{in}$ and $\mathcal{M}_{out}$ also hold [3]

$$K_\theta^{\theta-}\Big|_{\Sigma_1} = \frac{1}{R_1}\sqrt{1+\dot{R}_1^2} \quad , \quad K_\theta^{\theta+}\Big|_{\Sigma_2} = \frac{1}{R_2}\sqrt{1+\dot{R}_2^2 - \frac{\mathcal{R}(r_2)}{R_2}} \quad , \tag{12}$$

where $\mathcal{R}(r_2)$ is the Schwarzschild radius of the spherical shell within the comoving surface r_2. Now, to obtain the dynamical equation of the thick shell, we first expand the following quantities in a Taylor series around r_0, the mean comoving radius of the thick shell:

$$R(r_j, \tau) = R(r_0, \tau) + \epsilon_j \delta R'(r_0, \tau) + \mathcal{O}(\delta^2), \tag{13}$$

$$F(r_j) = F(R_0) + \epsilon_j \delta F'(r_0) + \mathcal{O}(\delta^2), \tag{14}$$

$$\mathcal{R}(r_2) = \mathcal{R}(r_0) + \delta \mathcal{R}'(r_0) + \mathcal{O}(\delta^2), \tag{15}$$

where $\epsilon_1 = -1$ and $\epsilon_2 = +1$. Using Eqs. (13), (14) and (15) in the expressions (11) and (12) and keeping only terms up to the first order of δ, we obtain

$$K_\theta^{\theta-}\Big|_{\Sigma_1} = \frac{1}{R_0}\sqrt{1+\dot{R}_0^2}\left(1+\delta\Big(\frac{R_0'}{R_0} - \frac{\dot{R}_0\dot{R}_0'}{1+\dot{R}_0^2}\Big)\right) \quad , \tag{16}$$

$$K_\theta^{\theta+}\Big|_{\Sigma_2} = \frac{1}{R_0}\sqrt{1+\dot{R}_0^2 - \frac{\mathcal{R}(r_0)}{R_0}}\left(1-\delta\Big(\frac{R_0'}{R_0} - \frac{\dot{R}_0\dot{R}'_0 - \frac{\mathcal{R}'(r_0)}{2R_0} + \frac{R_0'\mathcal{R}(r_0)}{2R_0^2}}{1+\dot{R}_0^2 - \frac{\mathcal{R}(r_0)}{R_0}}\Big)\right) \quad ,$$

$$K_\theta^{\theta+}\Big|_{\Sigma_1} = \frac{1}{R_0}\sqrt{1+\dot{R}_0^2 - \frac{F(r_0)}{R_0}}\left(1+\delta\Big(\frac{R_0'}{R_0} - \frac{\dot{R}_0\dot{R}'_0 - \frac{F'(r_0)}{2R_0} + \frac{R_0'F(r_0)}{2R_0^2}}{1+\dot{R}_0^2 - \frac{F(r_0)}{R_0}}\Big)\right) \quad ,$$

$$K_\theta^{\theta-}\Big|_{\Sigma_2} = \frac{1}{R_0}\sqrt{1+\dot{R}_0^2 - \frac{F(r_0)}{R_0}}\left(1-\delta\Big(\frac{R_0'}{R_0} - \frac{\dot{R}_0\dot{R}'_0 - \frac{F'(r_0)}{2R_0} + \frac{R_0'F(r_0)}{2R_0^2}}{1+\dot{R}_0^2 - \frac{F(r_0)}{R_0}}\Big)\right) \quad ,$$

where $R_0 \equiv R(r_0, \tau)$. Substituting Eq. (16) into Eq. (4) and noting that for the metric LTB, $F(r_0)$ plays just the role of the Schwarzschild radius of the part of the spherical thick shell within the comoving surface r_0 denoted by $\mathcal{R}(r_0)$, we obtain after some rearrangement the dust thick shell's equation of motion written up to the first-order in δ:

$$\alpha - \beta = 2\delta \frac{F'(r_0)}{2R_0\sqrt{1+\dot{R}_0^2 - \frac{F(r_0)}{R_0}}}$$

$$-\delta\left(\frac{R_0'}{R_0}(\alpha-\beta) + \dot{R}\dot{R'}_0\Big(\frac{\alpha-\beta}{\alpha\beta}\Big) + \frac{1}{2\beta R_0}\Big(\mathcal{R}'(r_0) + \frac{R_0'}{R_0}\mathcal{R}(r_0)\Big)\right), \tag{17}$$

with

$$\alpha \equiv \sqrt{1+\dot{R}_0^2} \quad , \quad \beta = \sqrt{1+\dot{R}_0^2 - \frac{\mathcal{R}(r_0)}{R_0}}. \tag{18}$$

This is the generalization of thin shell dynamical equation up to the first order of the thickness. It is now interesting to verify the thin shell limit of this thick shell dynamical equation. In order to do this, consider the following definition for the surface energy density of the infinitely thin shell[1]:

$$\sigma = \int_{-\epsilon}^{\epsilon} \rho(r,\tau)dn, \tag{19}$$

where n is the proper distance in the direction of the normal n_μ and 2ϵ is the physical thickness of the shell. With the metric (5), Eq. (19) takes the form

$$\sigma = \int_{-\delta}^{\delta} \rho(r,\tau)\frac{R'(r,\tau)}{\sqrt{1+E(r)}}dr. \tag{20}$$

Using Eqs. (6) and (7), we find that Eq. (20) can be written as

$$8\pi G\sigma = \int_{-\delta}^{\delta} \frac{F'(r)}{R^2\sqrt{1+\dot{R}^2 - \frac{F(r)}{R}}}dr\cdot \tag{21}$$

Using Eqs. (13) and (14), we may integrate Eq. (21) up to the first order in δ to get

$$8\pi G\sigma = 2\delta\frac{F'(r_0)}{R_0^2\sqrt{1+\dot{R}_0^2 - \frac{F(r_0)}{R_0}}} + \mathcal{O}(\delta^2). \tag{22}$$

Substituting (22) into Eq. (17), we get the following result

$$\alpha - \beta = 4\pi G\sigma R_0$$

$$-\delta\left(\frac{R_0'}{R_0}(\alpha-\beta) + \dot{R}_0\dot{R'}_0(\frac{\alpha-\beta}{\alpha\beta}) + \frac{1}{2\beta R_0}(\mathcal{R}'(r_0) + \frac{R_0'}{R_0}\mathcal{R}(r_0))\right). \tag{23}$$

Note that in the zero thickness limit of the shell, as $\delta \longrightarrow 0$, the second term on the right hand side of Eq. (23) is regular and goes to zero such that Eq. (23) reduces to the Israel's equation of motion for the dust thin shell in vacuum, as given by Israel [1], Ipser-Sikivie[7], Sato[8], and Khorrami-Mansouri [9].
To see the explicit effect of the thickness on the dynamics of the thick shell we rewrite Eq. (23) as

$$\alpha - \beta = 4\pi G\tilde{\sigma}R_0, \tag{24}$$

where $\tilde{\sigma}$, the effective surface density, is defined by

$$\tilde{\sigma} = \sigma - \frac{\delta}{4\pi G R_0}\left(\frac{R'_0}{R_0}(\alpha-\beta) + \dot{R}_0\dot{R}'_0\Big(\frac{\alpha-\beta}{\alpha\beta}\Big) + \frac{1}{2\beta R_0}\Big(\mathcal{R}'(r_0) + \frac{R'_0}{R_0}\mathcal{R}(r_0)\Big)\right). \tag{25}$$

We see that Eq. (24) has the same form as the well-known Israel's equation for a thin shell with the effective surface density $\tilde{\sigma}$. Now let us have a closer look at the terms within the brackets. Note that for a dust shell starting its collapse at rest, the velocity $\dot{R}_0$ is negative during the collapse, it also becomes more negative with r so that $\dot{R}'_0 < 0$, so the combination of $\dot{R}_0\dot{R}'_0$ must be positive. On the other hand, the Schwarzschild radius of the shell layers is increased with r so that $\mathcal{R}'(r_0) > 0$. Therefore all the terms within the bracket on the right hand side of Eq. (25) are positive. This leads to the result $\tilde{\sigma} < \sigma$. Now, solving Eq. (24) for $\dot{R}^2$ we find

$$\dot{R}_0^2 = 4\pi^2 G^2 R_0^2\tilde{\sigma}^2 + \frac{\mathcal{R}^2(r_0)}{64\pi^2 G^2 R_0^4\tilde{\sigma}^2} + \frac{\mathcal{R}(r_0)}{2R_0} - 1. \tag{26}$$

From Eq. (26), it follows that for a given R_0 and $\mathcal{R}(r_0)$ as long as $R_0 > \mathcal{R}(r_0)$, smaller $\tilde{\sigma}(\tilde{\sigma} < \sigma)$ leads to a larger $\dot{R}^2$. Therefore the first-order thickness corrections to the Israel thin shell approximation speed up the collapse of dust shell in vacuum.

IV. Conclusion

We have presented a simple procedure to investigate the dynamics of a spherical thick shell embedded in an otherwise spherically symmetric space-time, based on the Darmois conditions satisfying on the shell boundaries. As the simplest nontrivial example, we applied our scheme to the case of the collapse of a thick shell of dust in vacuum and obtained the zero thickness limit of our formalism which is just the Israel thin shell equation. It has been shown that the effect of thickness, up to the first order in the shell thickness, is to speed up its collapse in vacuum.

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